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The diffraction of tides by a narrow channel

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A Green's function for a semi-infinite rotating ocean of uniform depth is obtained, and the resulting near and far fields are estimated asymptotically.

Given a tide of uniform height at the mouth of a narrow channel on a semi-infinite ocean, the Green's function is used to calculate the diffracted Kelvin and Poincaré waves propagating up the channel and into the ocean.

1. Introduction

Assuming a time factor $e^{i\omega t}$, the linearized equations of motion of long waves in a sheet of water of uniform depth h, rotating about a vertical axis with angular velocity $\frac{1}{2}f$, can be reduced to

$$hk^2u = i\omega\zeta_x + f\zeta_y, \tag{1.1}$$

$$hk^2v = -f\zeta_x + i\omega\zeta_y, \tag{1.2}$$

$$\zeta_{xx} + \zeta_{yy} + k^2\zeta = 0. \tag{1.3}$$

In these equations, $u(x, y)$, $v(x, y)$ are the particle velocities, averaged over the depth, in the (right-handed) x, y directions, respectively, $h+\zeta$ is the depth in the disturbed state, the suffixes indicate partial derivatives with respect to x, y, and

$$k^2 = (\omega^2 - f^2)/\alpha^2, \tag{1.4}$$

where $\alpha = (gh)^{\frac{1}{2}}$ is the velocity of long waves when $f = 0$.

In an infinite sheet of water, plane waves of the form

$$\zeta = e^{-i(kx-\omega t)} \tag{1.5}$$

are possible provided $\omega > f$, with phase velocities ω/k. If there is an infinite barrier at (say) $x = 0$, then in $x > 0$ there may be Kelvin waves of the form

$$\zeta = \exp\left[\frac{-fx}{\alpha} + i\omega\left(\frac{y}{\alpha} + t\right)\right], \tag{1.6}$$

travelling with speed α along the boundary in the negative y direction.

In this paper we consider a semi-infinite ocean in the region $x > 0$ in which the depth is h_2, and the appropriate constants and variables in this region are denoted by the suffix 2 (e.g. α_2, ζ_2, etc.). A channel of uniform depth h_1 occupies the region $|y| < b$, $x < 0$, and connects with the ocean on the segment $x = 0$, $|y| < b$, as in figure 1. Appropriate quantities in the channel region are denoted by the suffix 1.

We shall assume that the tidal problem in the ocean region is solved to the extent that there is a known solution ζ_0 of the inhomogeneous counterpart of (1.1) to (1.4), with $u_0 = 0$ at $x = 0$. It will also be assumed that the channel is narrow, so that fb/α_2, $\omega b/\alpha_2$ are small quantities, whence the number

$$Z = k_2 b \tag{1.7}$$

is sufficiently small to allow us to make the approximation that, near the mouth,

$$\zeta_0(x, y) = [H_0 + (H_1/\alpha_2)(i\omega y - fx)]\, e^{i\omega t} + O(Z^2), \tag{1.8}$$

where H_0, H_1 are constants, and $r = (x^2+y^2)^{\frac{1}{2}} = O(b)$. The problem to be considered in this paper is to find solutions ζ_1, ζ_2, of (1.1) to (1.4) which satisfy the

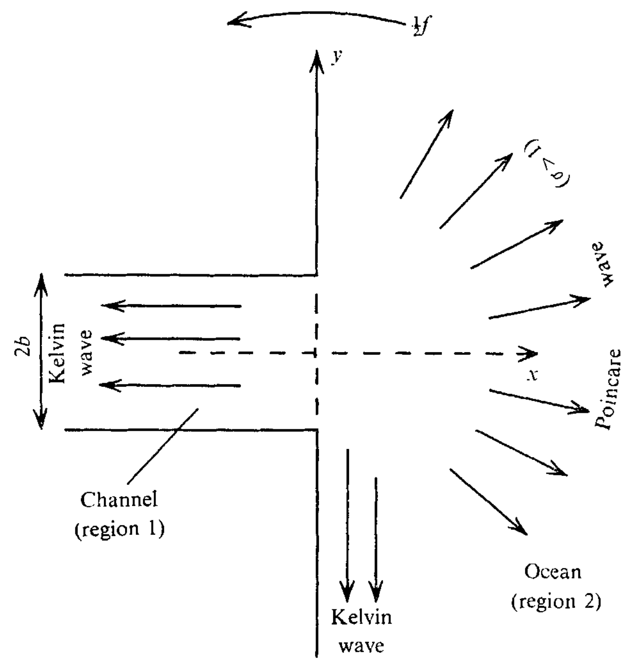

FIGURE 1. The waves due to diffraction by a narrow channel.

boundary conditions at the shore, and appropriate continuity conditions at the mouth of the channel, up to terms of $O(Z^2)$. It will be shown that, as illustrated in figure 1, the diffracted waves consist of Kelvin waves propagating along the coast and into the channel, together with cylindrical Poincaré waves radiating into the ocean provided $\omega > f$.

As a first step in the calculation, we shall obtain a Green's function for the ocean region in closed form, and asymptotically estimate the resulting near and far fields. The results for the far field agree with those of Voit (1958), while the near field reduces to the one obtained by Proudman (1925) in the special case $f^2/\omega^2 \ll 1$. The near-field solution derived in this paper is valid for all values of the ratio $\sigma = \omega/f$.

Once the Green's function is obtained and its properties are established, it is a fairly simple procedure to extend Proudman's (1925) method of matching ζ_1

and ζ_2 to all values of σ, and to evaluate in the case $h_1 = h_2$ the amplitudes and phases not only of the Kelvin wave proceeding up the channel, but also of the Kelvin waves on the coast, together with the Poincaré waves in the ocean.

If $h_1 \neq h_2$ there is no obvious way of applying the matching procedure, and in § 4 the exact solution of the diffraction problem is formulated as the solution of an integral equation. By resorting to a first-order Galerkin procedure and making use of the smallness of Z, an approximate solution is obtained. If $h_1 = h_2$ the latter solution gives the same amplitudes as Proudman's method, but there is a small error in the phases of the diffracted waves.

2. A Green's function for a semi-infinite ocean

Suppose that an ocean of uniform depth occupies the region $x > 0$, and that, except at the origin, there is a rigid barrier at $x = 0$. At the point (0, 0) we assume that there is an aperture producing an oscillating flow. In $x > 0$, ζ satisfies (1.3), and we assume that the boundary condition at $x = 0$ is

$$u(0, y) = \delta(y)\, e^{i\omega t}, \tag{2.1}$$

where $\delta(y)$ is the Dirac delta function. We shall also assume that ω has a small imaginary part, so that

$$\omega = \omega' - i\epsilon \quad (\epsilon > 0), \tag{2.2}$$

this being a convenient way of applying the radiation condition. The steady-state solutions may be obtained by taking $\epsilon \to 0+$.

An integral representation of ζ valid in $x \geqslant 0$ is

$$\zeta_G(x, y) = \int_{-\infty}^{\infty} \Phi(\beta)\, e^{-i\beta y - sx}\, d\beta, \tag{2.3}$$

and (1.3) is then satisfied if

$$s = (\beta^2 - k^2)^{\frac{1}{2}}. \tag{2.4}$$

The β plane is cut as in figure 2, and for convergence of the integral in (2.3) it is necessary to choose the branch for which $s \sim |\beta|$, as $\beta \to \pm\infty$.

By a simple application of Fourier's inversion theorem to (2.1), it is easily shown that

$$\Phi(\beta) = ihk^2/2\pi(f\beta + \omega s). \tag{2.5}$$

After rewriting (2.3) and (2.5) in the form

$$\zeta_G(x, y) = \frac{ih}{2\pi\alpha^2}\int_{-\infty}^{\infty} \left(\frac{\omega}{s} + \frac{\omega f^2}{\alpha^2 s(\beta^2 - \beta_0^2)} - \frac{f\beta}{\beta^2 - \beta_0^2}\right) e^{-i\beta y - sx}\, d\beta, \tag{2.6}$$

where $\beta_0 = \omega/\alpha$, transformations of standard forms (Campbell & Foster 1948, pairs 444, 445, 867, 868) can be used to show that, for $\sigma = \omega/f > 1$,

$$\zeta_G(x, y) = \sum_{j=1}^{3} B_j(x, y), \tag{2.7}$$

where
$$B_1(x,y) = (h\omega/2\alpha^2)\, H_0^{(2)}(kr), \tag{2.8}$$
$$B_2(x,y) = \frac{-ihf^2}{4\alpha^3}\int_{-\infty}^{\infty} H_0^{(2)}[k(x^2+\eta^2)^{\frac{1}{2}}]\, e^{-i\beta_0|y-\eta|}\, d\eta, \tag{2.9}$$
$$B_3(x,y) = \frac{ihfkx}{4\alpha^2}\int_{-\infty}^{\infty} \frac{H_1^{(2)}[k(x^2+\eta^2)^{\frac{1}{2}}]}{(x^2+\eta^2)^{\frac{1}{2}}}\, e^{-i\beta_0|y-\eta|}\, \mathrm{sgn}\,(y-\eta)\, d\eta, \tag{2.10}$$
$$B_3(0,y) = (-fh/2\alpha^2)\, e^{-i\beta_0|y|}\, \mathrm{sgn}\, y, \tag{2.11}$$
where $H_n^{(2)}$ are Hankel functions, and $r = (x^2+y^2)^{\frac{1}{2}}$. If $\sigma < 1$, we write $k = -ik'$ and the above expression may be written in terms of modified Bessel functions, using the rule $K_n(z) = -\frac{1}{2}\pi i\, e^{-\frac{1}{2}n\pi i} H_n^{(2)}(-iz)$.

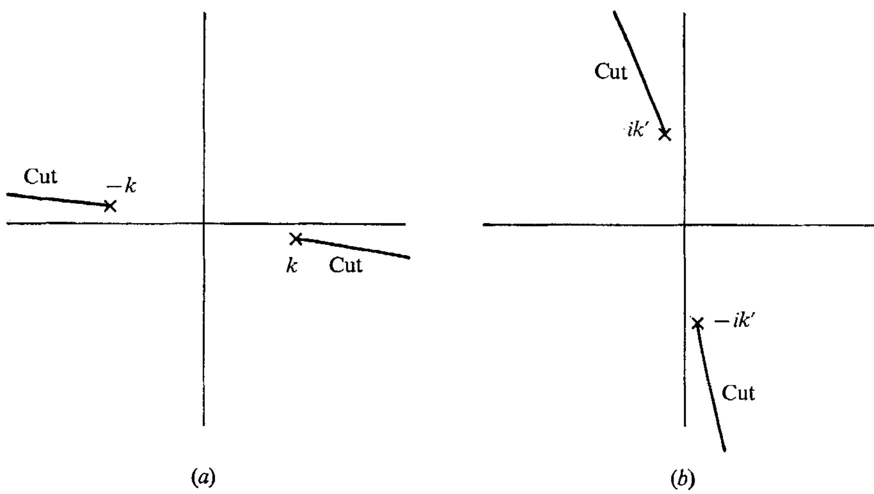

FIGURE 2. Cuts in complex plane for (a) $\sigma > 1$, and (b) $\sigma < 1$.

It should be noticed that in (2.8)–(2.10), $B_1(x,y)$ is the well-known solution in the case $f = 0$. Moreover, $B_2(x,y)$ is symmetric in y and $B_3(x,y)$ is antisymmetric, and each contains one half of the Kelvin wave. Both B_2 and B_3 also contain terms representing an edge wave propagating in the positive y direction, but these terms cancel in the sum in (2.7).

Although ζ_G is given explicitly by either (2.3) or (2.7), the following asymptotic approximations are useful.

(a) The far field

Expressions for the far field have been obtained by Voit (1958), and the elements of a more concise derivation are given below. Seshadri (1962) and Williams (1964) have also obtained similar results, in a different physical context limited to positive values of k^2.

(i) $\sigma > 1$. The transformations
$$x = r\cos\theta, \quad y = r\sin\theta, \quad \beta = k\sin\rho, \quad s = ik\cos\rho, \tag{2.12}$$
where $\rho = \mu + i\nu$, are applied to the integral in (2.3), with the result
$$\zeta_G = \frac{ihk^2}{2\pi}\int_L \frac{\cos\rho}{f\sin\rho + i\omega\cos\rho}\exp\left[-ikr\cos(\theta-\rho)\right] d\rho, \tag{2.13}$$

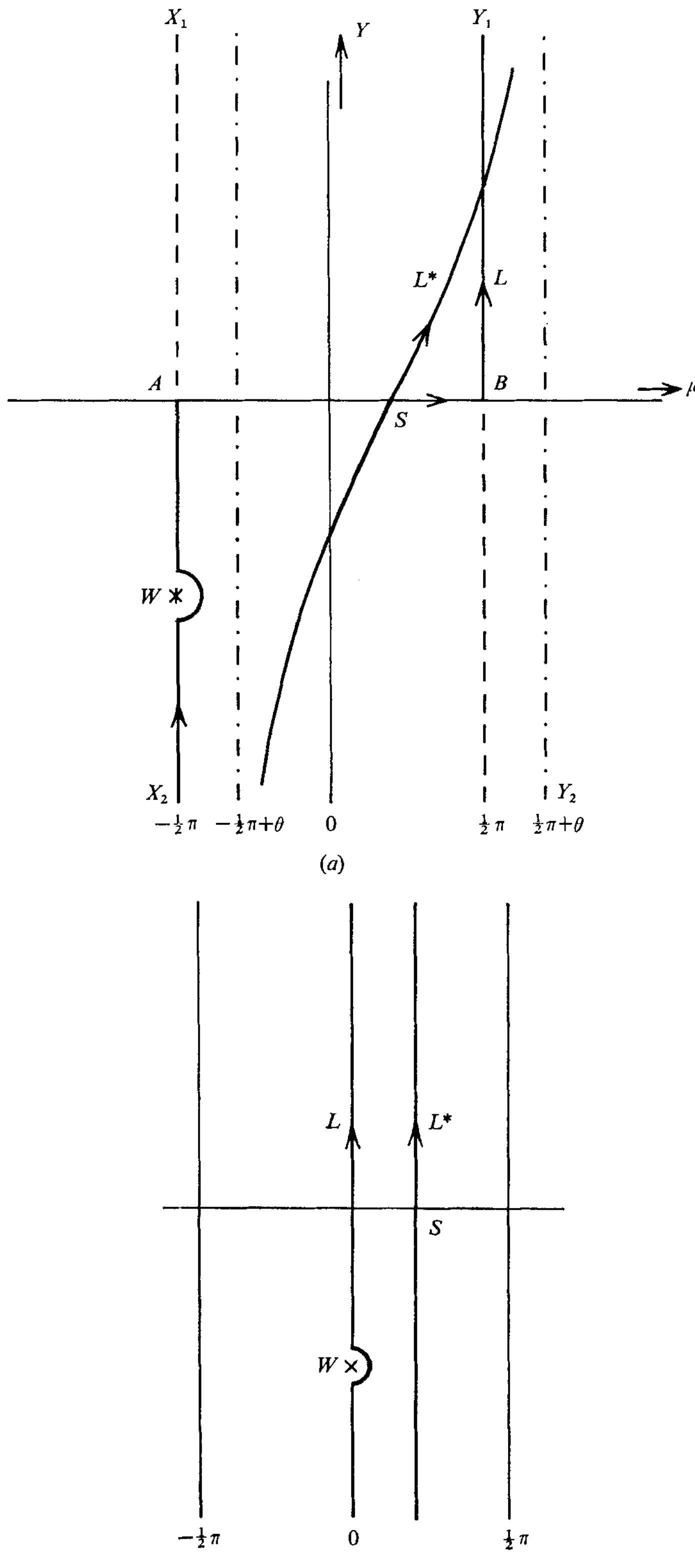

FIGURE 3. The path of integration L and the path of steepest descent L^* in the complex ρ plane for (a) $\sigma > 1$, and (b) $\sigma < 1$.

the path of integration L being the segments X_2ABY_1 in figure 3(a). The pole at $\beta = -\beta_0$ is transformed into a pole at W, where $\rho = \frac{1}{2}\pi - i\sin^{-1}(\beta_0/k)$, and an application of the radiation condition shows that L is to the right of W.

The saddle point is at $\rho = \theta$, and the path of integration L is deformed to the path L^*, through the saddle point, given by $\cos(\mu-\theta)\cosh\nu = 1$, as in figure 3($a$).

The pole is captured if $\theta < -\sin^{-1}(k/\beta_0)$, and, using standard results for the method of stationary phase, we obtain

$$\zeta_G = K_P + K_K H(\theta_0 - \theta), \tag{2.14}$$

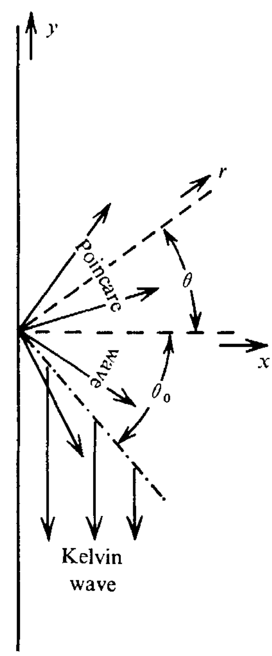

FIGURE 4. The far field for the Green's function $\zeta_G(x, y)$.

where $H(z)$ is the Heaviside unit function, $\theta_0 = -\sin^{-1}(k/\beta_0)$; and

$$K_P = hk\left(\frac{k}{2\pi r}\right)^{\frac{1}{2}} \frac{\cos\theta}{\omega\cos\theta - if\sin\theta} e^{-i(kr-\frac{1}{4}\pi)} + O(r^{-\frac{3}{2}}) \tag{2.15}$$

is a cylindrical wave radiating from the origin, while

$$K_K = (f/g)\, e^{(i\omega y - fx)/\alpha} \tag{2.16}$$

represents a Kelvin wave travelling along the negative y axis, as in figure 4.

The result in (2.14)–(2.16) agree with those of Voit (1958) (Voit's equations (2.8) and (2.9), after allowance has been made for misprints and notation changes).

(ii) $\sigma < 1$. In this case

$$\beta = -ik'\sin\rho, \quad s = k'\cos\rho, \tag{2.17}$$

where $k' = ik = (f^2-\omega^2)^{\frac{1}{2}}/\alpha$. The paths of integration L and L^* are as in figure 3(b), the saddle point being at $\rho = \theta$. Using the method of steepest descent, the pole is captured only for $\theta < 0$, with the result that

$$\zeta_G = K_P + K_K H(-\theta), \tag{2.18}$$

where, in this case,

$$K_P = \frac{hk'\cos\theta}{i\omega\cos\theta + f\sin\theta}\left(\frac{k'}{2\pi r}\right)^{\frac{1}{2}} e^{-k'r} + O(e^{-k'r}r^{-\frac{3}{2}}), \tag{2.19}$$

and K_K is given by (2.16). Note that only the Kelvin wave is propagated.

(b) *The near field*

Assuming $\sigma > 1$, it may be shown that for $|yk| \ll 1$,

$$B_1(0,y) = \frac{\omega}{2g}\left[1 - \frac{2i}{\pi}\log\tfrac{1}{2}\gamma k|y|\right] + O[(ky)^2\log y], \tag{2.20}$$

$$B_2(0,y) = \frac{f}{2\pi i g}\log\frac{\sigma+1}{\sigma-1} + O[(ky)^2\log y] \tag{2.21}$$

$$B_3(0,y) = -\frac{f}{2g}[1 - i\beta_0|y|]\operatorname{sgn} y + O[(ky)^2], \tag{2.22}$$

where $\log\gamma$ is Euler's constant, and the result (Erdelyi *et al.* 1954)

$$\int_0^\infty H_0^{(2)}(k\eta)\, e^{-p\eta}\, d\eta = \frac{1}{(p^2+k^2)^{\frac{1}{2}}}\left[1 + \frac{2i}{\pi}\sinh^{-1}\frac{p}{k}\right]$$

has been used.

If $x \neq 0$ the derivation is not quite so straightforward, but it may be shown that if $\sigma > 1$

$$\zeta_G(r,\theta) = \frac{\omega}{2g}\left[1 - \frac{2i}{\pi}\log\tfrac{1}{2}\gamma kr - \frac{1}{\pi\sigma}\left(2\theta + i\log\frac{\sigma+1}{\sigma-1}\right)\right] + O(kr). \tag{2.23}$$

If $\sigma < 1$, then $k = -ik'$, when

$$\zeta_G(r,\theta) = \frac{f}{2g}\left[1 - \frac{2i\sigma}{\pi}\log\tfrac{1}{2}\gamma k'r - \frac{1}{\pi}\left(2\theta + i\log\frac{1+\sigma}{1-\sigma}\right)\right] + O(k'r). \tag{2.24}$$

The formulae in (2.23) and (2.24) give the near field of ζ_G uniformly for all values of σ. If $\sigma = 1$,

$$\zeta_G = \frac{f}{2g}\left[1 - \frac{2i}{\pi}\left(\log\frac{\gamma fr}{\alpha} - i\theta\right)\right] + O(fr/\alpha).$$

3. Propagation into a narrow channel

The geometry of figure 1, with $h_1 = h_2$, is assumed. Following Proudman (1925) we take two outer regions in the channel, and ocean, respectively, and an inner matching region at the mouth of the channel, where $r = O(b)$.

The displacement in the channel can be expressed as the series

$$\zeta_1(x,y) = A_0\, e^{(fy+i\omega x)/\alpha} + \sum_{n=1}^{\infty} A_n[f\beta_n \sin\mu_n(y+b) + \omega\mu_n\cos\mu_n(y+b)]\, e^{i\beta_n x}, \tag{3.1}$$

where $\mu_n = n\pi/2b, \qquad \beta_n^2 = k^2 - \mu_n^2,$ (3.2)

and the time factor $e^{i\omega t}$ is assumed. Note that unnecessary suffices have been dropped. The A_n are arbitrary constants, and it is easily shown from (1.2) that $v_1(x, \pm b) = 0$. Also, substitution in (1.1) yields

$$hu_1(0,y) = -\alpha A_0\, e^{fy/\alpha} - \sum_{n=1}^{\infty} A_n[\omega f\sin\mu_n(y+b) + \alpha^2\beta_n\mu_n\cos\mu_n(y+b)]. \tag{3.3}$$

If $bk < \frac{1}{2}\pi$, then β_n is negative imaginary, and only the Kelvin wave, represented by the term in A_0 in (3.1), gives waves which propagate along the channel.

If it assumed that $Z = kb \ll 1$, but the ratios $f:\omega:\alpha k$ are $O(1)$, then

$$\beta_n = -i\mu_n + O(Z^2).$$

Near the channel end, where $r = O(b)$, we may rewrite (3.1) in the form

$$\zeta_1(x,y) = A_0[1 + (fy + i\omega x)/\alpha + O(Z^2)] + \sum_{n=1}^{\infty} \mu_n A_n [\omega \cos \mu_n(y+b) - if \sin \mu_n(y+b)]\, e^{\mu_n x}. \tag{3.4}$$

In the ocean region it is assumed that for $x > 0$, and $r = O(b)$,

$$\zeta_2(x,y) = \zeta_0(x,y) + D\zeta_G(x,y), \tag{3.5}$$

where ζ_0 is given in (1.8), ζ_G is given in (2.7), and D is a constant which is to be determined. Near the channel ζ_G is given by (2.23) or (2.24).

In the matching region we write

$$\xi = x/b, \quad \eta = y/b, \tag{3.6}$$

when, in the new co-ordinates,

$$\nabla^2\zeta + Z^2\zeta = 0,$$

so that, in this region,

$$\zeta = \zeta_M + O(Z^2), \tag{3.7}$$

where ζ_M satisfies Laplace's equation. It can now be shown that the boundary conditions and the conditions at infinity may be satisfied if ζ_M is expressed in the form

$$\zeta_M = B + \sum_{i=1}^{2} C_i(i\omega\phi_i + f\psi_i), \tag{3.8}$$

where ϕ_i, ψ_i, are conjugate harmonic functions determined by the mappings

$$w_1 = \phi_1 + i\psi_1 = 2\log\tau - i\pi, \tag{3.9}$$

$$w_2 = \phi_2 + i\psi_2 = \tau, \tag{3.10}$$

$$z = \pi(\xi + i\eta) = 2\log\tau - 2\log[(\tau^2-1)^{\frac{1}{2}} + i] - 2i(\tau^2-1)^{\frac{1}{2}}, \tag{3.11}$$

where, in (3.11), the branch is chosen so that $(\tau^2-1)^{\frac{1}{2}} \sim \tau$, as $\tau \to \infty$.

In the channel, as $\tau \to 0$, $\xi \to -\infty$, and it can be shown that the following asymptotic expansions hold:

$$w_1 = \pi(x+iy)/b + 2\log 2 - 2 + \sum_{m=1}^{\infty} L_m\, e^{\mu_{2m}(x+iy)}, \tag{3.12}$$

$$w_2 = i \sum_{p=1}^{\infty} M_p\, e^{\mu_{2p-1}(x+iy)}, \tag{3.13}$$

where $M_1 = 2e^{-1}$, and the other real coefficients L_m, M_p, are obtainable from (3.9) to (3.11) by successive numerical approximation.

In the ocean, as $\tau \to \infty$, $z \to \infty$, $|\arg z| \leq \frac{1}{2}\pi$, when

$$w_1 = 2\log(\pi r/2b) + 2i\theta + O[(r/b)^{-2}], \tag{3.14}$$

$$w_2 = \pi i(x+iy)/2b + O[b/r], \tag{3.15}$$

where $r = (x^2+y^2)^{\frac{1}{2}}$.

The next step is to match ζ_M as $x \to -\infty$ with ζ_1 as $x \to 0$, and ζ_M as $r \to \infty$ with ζ_2 as $r \to 0$. After some algebra, we find that, for $\sigma > 1$,

$$H_0 = A_0\left\{1 + \frac{2\omega bi}{\pi\alpha}\left(1 - \log\frac{2\gamma bk}{\pi} - \tfrac{1}{2}\pi i - \frac{1}{2\sigma}\log\frac{\sigma+1}{\sigma-1}\right) + O(Z^2)\right\}, \tag{3.16}$$

and

$$\alpha D = -2gbA_0. \tag{3.17}$$

If $\sigma < 1$,

$$H_0 = A_0\left\{1 + \frac{2fbi}{\pi\alpha}\left(\sigma - \sigma\log\frac{2\gamma bk'}{\pi} - \tfrac{1}{2}\pi i - \tfrac{1}{2}\log\frac{1+\sigma}{1-\sigma}\right) + O(Z^2)\right\}. \tag{3.18}$$

These results are valid for all values of σ, there being no singularity at $\sigma = 1$. If it is assumed that σ^{-2} is small enough to be neglected when compared with unity, then Proudman's (1925) results are obtainable from (3.16). It should be noticed that by virtue of (3.17), the coefficient of the second term of (3.5) is $O(Z)$, and hence, although the error in ζ_G in (2.23) and (2.24) is $O(Z)$ as $r \to 0$, equation (3.5) has errors of only $O(Z^2)$. Another consequence is that we may replace A_0 by H_0 in (3.17) without loss of accuracy.

It is easily seen from (3.16) and (3.18) that

$$\begin{aligned} |A_0/H_0| &= 1-(\omega b/\alpha) \quad (\sigma \geqslant 1), \\ &= 1-(fb/\alpha) \quad (\sigma \leqslant 1), \end{aligned} \tag{3.19}$$

if terms of $O[(Z\log Z)^2]$ are neglected. There is also a change of phase which is $O[Z\log Z]$.

The coefficients of the diffracted Kelvin and Poincaré waves are determined from (2.15), (2.16), (3.5), (3.17) and (3.19), with the result that

$$\zeta_K = -\frac{2fb}{\alpha} H_0 e^{(i\omega y - fx)/\alpha}, \tag{3.20}$$

$$\zeta_P = -2kb\alpha H_0\left(\frac{k}{2\alpha r}\right)^{\frac{1}{2}} \frac{\cos\theta}{\omega\cos\theta - if\sin\theta} e^{-i(kr-\frac{1}{4}\pi)} \quad (\sigma > 1), \tag{3.21}$$

and ζ_K, ζ_P are the diffracted Kelvin and Poincaré waves, respectively. Note that the amplitude of the diffracted Kelvin wave is $O(Z)$, and that the phase is reversed when compared with the incident wave.

The coefficients in the sum in (3.1) are also obtained in the matching procedure in the form

$$\mu_{2p}A_{2p} = \frac{ib}{\alpha\pi}(-1)^p A_0 L_p + O(Z^2), \tag{3.22}$$

$$\mu_{2p-1}A_{2p-1} = \frac{2ib}{\alpha\pi}(-1)^p H_1 M_p + O(Z). \tag{3.23}$$

The term in H_1 in (1.8) does not contribute to the diffracted waves to the order of magnitudes considered.

4. The effect of a change in depth

If $h_1 \neq h_2$ it is not easy to see how to use the conformal mapping method, and an alternative procedure is adopted. The continuity conditions at the mouth of the channel are

$$\zeta_1 = \zeta_2, \quad h_1 u_1 = h_2 u_2, \tag{4.1}$$

when $x = 0$ and $|y| < b$. Otherwise the geometry and other conditions are as in § 3. Now suppose that ζ_1 is known in the channel, where ζ_1 is given by the series (3.1), except that $\alpha_1 = (gh_1)^{\frac{1}{2}}$, and $k_1^2 = (\omega^2 - f^2)/\alpha_1^2$, so that

$$\beta_n^2 = k_1^2 - u_n^2.$$

Then $h_1 u_1(0, y)$ is given by the series (3.3), and using (4.1) together with the Green's function, we obtain the integral equation

$$\zeta_1(0, y) = H_0 + \Delta \int_{-b}^{b} u_1(0, \eta)\, \zeta_G(0, y-\eta)\, d\eta, \tag{4.2}$$

for $|y| \leqslant b$, where ζ_G is given by (2.7)–(2.11) and $\Delta = h_1/h_2$. Using § 3 as a guide, it has been assumed that the term in H_1 in (1.8) does not give rise to propagating diffracted waves, and may be disregarded for this reason. As a first approximation we use a first-order Galerkin technique, i.e. we approximate ζ_1 and u_1 by the term in A_0 only, multiply (4.2) by e^{fy/α_1} and integrate from $-y$ to y. Since the channel is narrow we use the near-field approximation for $\sigma > 1$,

$$\zeta_G(0, y) = \frac{f}{2g}\left[\sigma\left(1 - \frac{2i}{\pi}\log \tfrac{1}{2}k_2\gamma|y|\right) - \frac{i}{\pi}\log\frac{\sigma+1}{\sigma-1} - \operatorname{sgn} y\right],$$

to obtain, after some computation, that

$$\begin{aligned}\frac{2H_0 \sinh F}{A_0} = \sinh 2F + \frac{2\Delta\sigma i}{\pi}\Big[&\sinh F \operatorname{shi} F - \cosh F \operatorname{cinh} F \\ &- \sinh^2 F\left(\pi i + \sigma^{-1}\log\frac{\sigma+1}{\sigma-1} + 2\log \gamma b k_2\right)\Big],\end{aligned} \tag{4.3}$$

where $\Delta = h_1/h_2$, $F = fb/\alpha_1$, and

$$\operatorname{shi} F = \int_0^F \frac{\sinh z}{z}\, dz, \quad \operatorname{cinh} F = \int_0^F (\cosh z - 1)\frac{dz}{z}.$$

Noting that as $F \to 0$, $\operatorname{shi} F \sim F$, and $\operatorname{cinh} F \sim \frac{1}{4}F^2$, it may be shown for a narrow channel that, for $\sigma > 1$,

$$H_0 = A_0\left\{1 + \frac{2\omega b \Delta i}{\pi\alpha_1}\left(\tfrac{3}{2} - \log \gamma b k_2 - \tfrac{1}{2}\pi i - \frac{1}{2\sigma}\log\frac{\sigma+1}{\sigma-1}\right)\right\} \quad (\sigma > 1), \tag{4.4}$$

with a similar result for $\sigma < 1$, that

$$H_0 = A_0\left\{1 + \frac{2fb\Delta i}{\pi\alpha_1}\left(\tfrac{3}{2}\sigma - \sigma\log \gamma b k_2' - \tfrac{1}{2}\pi i - \tfrac{1}{2}\log\frac{1+\sigma}{1+\sigma}\right)\right\}. \tag{4.5}$$

It follows from (4.4) and (4.5) that up the channel

$$\begin{aligned}|A_0/H_0| &= 1 - (\omega\Delta b/\alpha_1), \quad (\sigma \geqslant 1), \\ &= 1 - (f\Delta b/\alpha_1), \quad (\sigma \leqslant 1).\end{aligned} \tag{4.6}$$

If the channel and ocean regions are of the same depth, then $\Delta = 1$, and the formulae in (4.6) are identical with (3.19). Note also that the inner brackets in (4.4) and (4.5) differ from the inner brackets in (3.16) and (3.18) by

$$\tfrac{1}{2} - \log \tfrac{1}{2}\pi \approx 0{\cdot}05,$$

so that the errors in phase in A_0 in (4.4) and (4.5) are very small, being about $Z/10\pi$. Assuming that this error may be neglected, the results obtained by the Galerkin technique for $\Delta = 1$ agree with those obtained by Proudman's method. It should be emphasized here that it is not certain that the Galerkin method should give accurate results, although when the method is used in similar circumstances (Miles & Munk 1961) good results are obtainable. However, in this case a direct comparison with § 3 in the case $\Delta = 1$ suggests that the results in (4.6) are correct up to $O(Z)$.

We have assumed that $h_1 u_1(0, y) = -\alpha_1 A_0 e^{fy/\alpha_1}$, so that by convolution with (2.16), and using (4.1), the diffracted Kelvin wave is given by

$$\zeta_K = -\frac{2\Delta fb}{\alpha_1} H_0 e^{(i\omega y - fx)/\alpha_2}, \tag{4.7}$$

for small Z, which is in agreement with (3.20) if $\Delta = 1$. Similarly, the diffracted Poincaré wave contains the factor Δ compared with (3.21).

5. Conclusion

It has been shown that a narrow channel on a straight coastline gives rise to diffracted Kelvin and Poincaré waves, the amplitudes of which are proportional to the width of the channel and to the ratio of the depth in the channel and the ocean. Munk, Snodgrass & Wimbush (1970) have made extensive theoretical and observational studies of the tides on the California coast, with results that indicate very significant Kelvin wave components in the observed tides. Bearing in mind that a slight modification of the results of Packham & Williams (1968) would show that a sharp bend in a coastline is an effect of conversion of tides into Kelvin waves, one would expect that a combination of bends, bays and channels would make a significant contribution to the generation of the Kelvin wave components of the observed tides.

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